Double Actions of Monoidal Categories

Accompanying notes for

'The 2-Dimensional Structure of Tambara Modules'

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(based on joint work with Matteo Capucci)

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In the following we assume $\mathbb{E}$ is a monoidal (pseudo)double category [4]. By a psuedomonoid object in $\mathbb{E}$ we mean a pseudomonoid[2] in the monoidal 2-category of tight arrows in $\mathbb{E}$. We use m to denote pseudomonoid multiplication morphisms, and e for unit morphisms, and α, λ, ρ for the structure 2-cells.

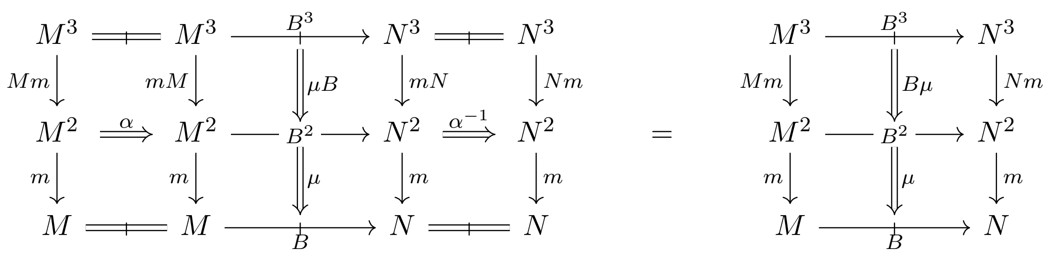
Viewing proarrows in $\mathbb{E}$ from the perspective of 'objects with boundaries' we define *monoidal proarrows* to be compatible monoid objects in the category of arrows, $\mathbb{E}_1$.

Definition 1 (Monoidal Proarrows)**.** Let $\mathbb{E}$ be a monoidal double category. If M and N are pseudomonoids in $\mathbb{E}$, then a monoidal proarrow $M \to N$ is a loose arrow $B : M \to N$ such that (B, μ, η) is a monoid object in $\mathbb{E}_1$

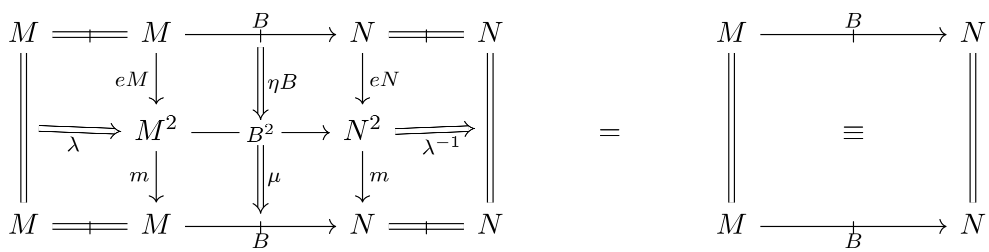
$$\begin{array}{ccc} M^2 & \xrightarrow{B^2} & N^2 \\ {\scriptstyle m}\downarrow & \Downarrow \mu & \downarrow{\scriptstyle m} \\ M & \xrightarrow[B]{} & N \end{array} \qquad\qquad \begin{array}{ccc} I & = & I \\ {\scriptstyle e}\downarrow & \Downarrow \eta & \downarrow{\scriptstyle e} \\ M & \xrightarrow[B]{} & N \end{array}$$

satisfying the following equations:

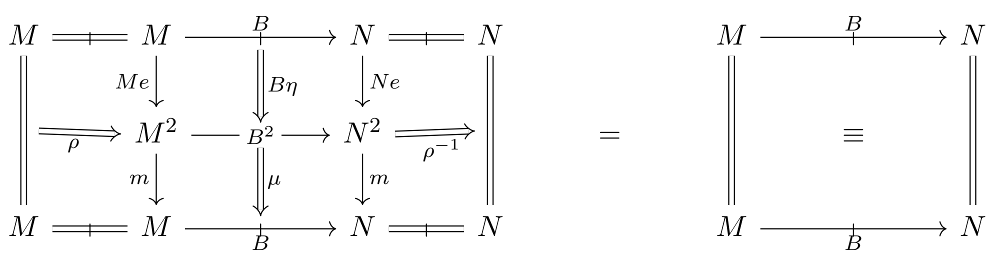
Compatibility with associators



Compatibility with left unitors



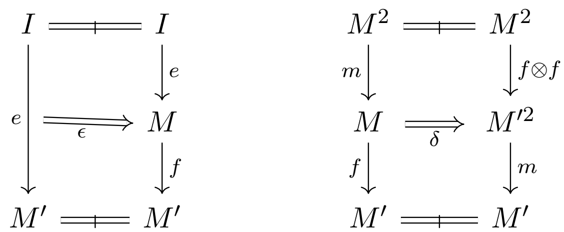
Compatibility with right unitors



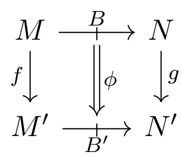
Remark 2. Here we have presented the definition of a monoidal proarrow as a kind of generalised monoid object, but alternatively we can consider it as describing a lax monoid-morphism between the corresponding pseudomonoids. In this view, the monoid multiplication for B becomes the lax cell witnessing that B is a homomorphism with respect to M's monoidal structure.

We choose here to use the terminology of B being itself a monoid object as this aligns more closely with the later applications, rather than viewing it as a 'map' itself. Additionally this makes it more obvious what the corresponding 2-cells between monoidal proarrows should be.

We can define a morphism of monoidal proarrows similarly to before, as a compatible monoid morphism in $\mathbb{E}_1$, bordered by pseudomonoid morphisms. For a pseudomonoid morphism $f : M \to M'$ we denote the unitality and distributivity isomorphisms ϵ and δ as below:

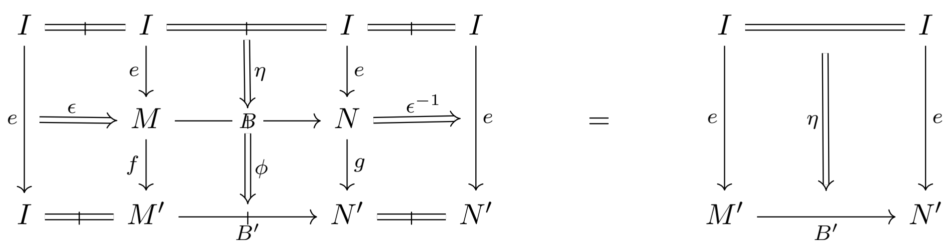


Definition 3 (Monoidal proarrow morphisms)**.** A morphism of monoidal proarrows is a square

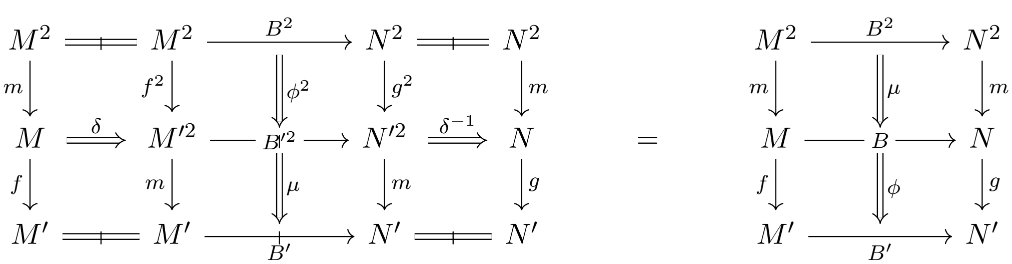


where f and g are (strong) pseudomonoid morphisms and ϕ is a monoid morphism, and such that ϕ satisfies similar compatibility equations:

Compatibility with unitality cells



Compatibility with distributivity cells



Example 4 (Monoidal Profunctors)**.** Our motivating example is that where $\mathbb{E} = \mathbb{P}\mathbf{rof}$. Then pseudomonoid objects are small monoidal categories and a monoidal proarrow $P : \mathcal{M} \to \mathcal{N}$ is a profunctor which is lax monoidal as a functor out of $\mathcal{M} \times \mathcal{N}^{\mathrm{op}}$:

$$1 \to P(I, I)$$

$$P(M, N) \times P(M', N') \to P(M \otimes M', N \otimes N')$$

and the compatibility equations amount to the associativity and unitality conditions of a monoidal functor.

The way in which we conjugate by the structure isomorphisms in the above equations in fact means that the composite of monoidal proarrows remains a monoidal proarrow. Hence we obtain a double category of monoidal proarrows:

Proposition 5. *For a monoidal double category* $\mathbb{E}$ *there is a double category* $\mathbf{Mnd}(\mathbb{E})$ *which has*

- *objects as pseudomonoids in* $\mathbb{E}$*,*
- *tight morphisms as strong monoid morphisms,*
- *loose morphisms as monoidal proarrows,*
- *square cells morphisms of monoidal proarrows.*

□

A pseudomonoid object acts on an object of the monoidal 2-category, giving the notion of a pseudomodule ($M \otimes X \xrightarrow{b} X$) [1]. We similarly define *modular proarrows* as those proarrows with an action of a monoidal proarrow.

Definition 6 (Modular Proarrows)**.** Let X be a left M-pseudomodule and Y a left N-pseudomodule and let $B : M \to N$ be a monoidal proarrow, then a left action of B on a proarrow $P : X \to Y$ is a monoid action of B on P in $\mathbb{E}_1$

$$\begin{array}{ccc} M \otimes X & \xrightarrow{B \otimes P} & N \otimes Y \\ {\scriptstyle b}\downarrow & \Downarrow{\scriptstyle \beta} & \downarrow{\scriptstyle b} \\ X & \xrightarrow[P]{} & Y \end{array}$$

compatible with the pseudomodule structures on M and N in a similar way to in definition 1 We call a proarrow P with such a structure a (left) modular proarrow.

Similarly we define right modular proarrow as right actions on proarrows between right pseudomodules. If an object X has left and right module structures compatible up to isomorphism

$$
\begin{array}{ccc}
M \otimes X \otimes M' & \overset{}{=\!\!\!\mid\!\!\!=} & M \otimes X \otimes M' \\
{\scriptstyle \text{left action}}\downarrow & & \downarrow{\scriptstyle \text{right action}} \\
X \otimes M' & \overset{\sim}{\Longrightarrow} & M \otimes X \\
{\scriptstyle \text{right action}}\downarrow & & \downarrow{\scriptstyle \text{left action}} \\
X & =\!\!\!=\!\!\!\mid\!\!\!=\!\!\!= & X
\end{array}
$$

we call it a pseudo-bimodule. In the case that X and Y are pseudo-bimodules we define a bimodular proarrow as a proarrow $P : X \to Y$ with compatible left and right module structures

$$
\begin{array}{ccccccc}
M \otimes X \otimes M' & =\!\!\!\mid\!\!\!= & M \otimes X \otimes M' & \xrightarrow{B \otimes P \otimes B'} & N \otimes Y \otimes N' & =\!\!\!\mid\!\!\!= & N \otimes Y \otimes N' \\
\downarrow & & \downarrow & \Downarrow & \downarrow & & \downarrow \\
X \otimes M' & \cong & M \otimes X & \xrightarrow{B \otimes P} & N \otimes Y & \cong & Y \otimes N' \\
\downarrow & & \downarrow & \Downarrow & \downarrow & & \downarrow \\
X & =\!\!\!=\!\!\!\mid\!\!\!=\!\!\!= & X & \xrightarrow[P]{} & Y & =\!\!\!=\!\!\!\mid\!\!\!=\!\!\!= & Y
\end{array}
$$

$$\parallel$$

$$
\begin{array}{ccc}
M \otimes X \otimes M' & \xrightarrow{B \otimes P \otimes B'} & N \otimes Y \otimes N' \\
\downarrow & \Downarrow & \downarrow \\
X \otimes M' & \xrightarrow{P \otimes B'} & Y \otimes N' \\
\downarrow & \Downarrow & \downarrow \\
X & \xrightarrow[P]{} & Y
\end{array}
$$

We similarly have that composites of bimodular proarrows are bimodular proarrows. Then defining morphisms of bimodular proarrows similarly to morphisms of monoidal proarrows we obtain another double category

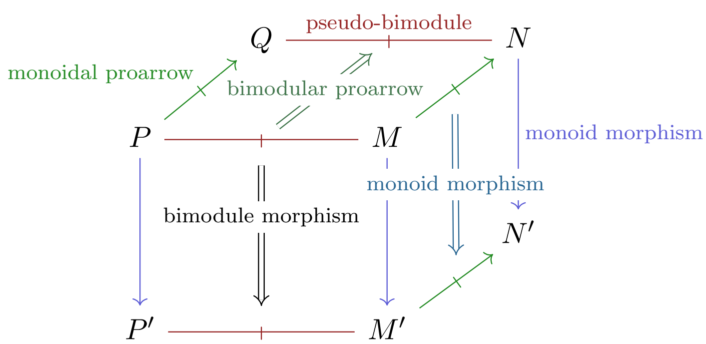
Proposition 7. *There exists a double category* $\mathbf{Bimod}(\mathbb{E})$ *consisting of*

- *objects are triples* (M, X, M') *where* M *and* M' *are pseudomonoids in* $\mathbb{E}$ *and* X *is an* (M, M')*-bimodule,*

- *tight morphisms are pseudomodule morphisms*
- *loose morphisms are bimodular proarrows*
- *square cells are morphisms of bimodular proarrows*

□

As well as this compositional structure we can consider another form of composition, which is that of bimodule composition. The objects of this double category themselves have a form of domain/codomain, given by the acting pseudomonoids. As is common with bimodule objects, we can compose these bimodules by quotienting out common actions when the category has enough colimits. This additional direction for composition gives rise to a cubical structure, where we have three classes of morphism, with cubical cells between them.



Conjecture 8. *If $\mathbb{E}$ has coequalisers and local coequalisers then there is a pseudocategory internal to* **DblCat** *whose double category of objects is* $\mathbf{Mnd}(\mathbb{E})$ *and whose double category of morphisms is* $\mathbf{Bimod}(\mathbb{E})$

When $\mathbb{E} = \mathbb{P}\mathbf{rof}$ we this construction has

- Objects are monoidal categories.
- The 1-cells are:
  - strong monoidal functors,
  - profunctors with lax monoidal structure,
  - and 2-sided actegories (bimodules of monoidal categories).
- The square cells are:

– natural transformations between monoidal profunctors,

– functors between actegories, linear in their actions,

– profunctors between actegories, with 2-sided actions of monoidal profunctors

$$P(M, M') \times Q(C, D) \to Q(M \bullet C, M' \bullet D).$$

- Cubical cells are natural transformations of actegory-profunctors compatible with their bordering morphisms in appropriate senses.

In this characterisation we recover Tambara's notion of two-sided action on a profunctor [3] as literal actions of identity profunctors.

Proposition 9. *Tambara modules*

$$\tau_M : Q(C, D) \to Q(M \bullet C, M \bullet D)$$

$$\sigma_M : Q(C, D) \to Q(C \bullet N, D \bullet N)$$

are equivalent to square profunctors which are globular in one direction:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad \mathcal{C} \quad} & \mathcal{N} \\ \Big\| & \overset{Q}{\Downarrow} & \Big\| \\ \mathcal{M} & \xrightarrow{\quad \mathcal{D} \quad} & \mathcal{N} \end{array}$$

□

Summary of notation

m	Pseudomonoid multiplication map
e	Pseudomonoid unit
b	Pseudomodule action
λ, ρ	Pseudomonoid unitor cells
α	Pseudomonoid associator cell
ϵ	Pseudomonoid morphism unitality cell
δ	Psueodmonoid morphism distributivity cell
μ	Monoid multiplication map
η	Monoid unit

References

[1] Matteo Capucci and Bruno Gavranović. Actegories for the working amthematician, 2022.

[2] Paddy McCrudden. Balanced coalgebroids, 2000.

[3] Craig Pastro and Ross Street. Doubles for monoidal categories, 2007.

[4] Michael A. Shulman. Constructing symmetric monoidal bicategories, 2010.