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The 2-Dimensional Structure of Tambara Modules

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Background

When a monoidal category $\mathcal{M}$ acts on a category $\mathcal{C}$ we obtain a notion of an $\mathcal{M}$-*strong* profunctor, referred to as a *Tambara module* [1]:

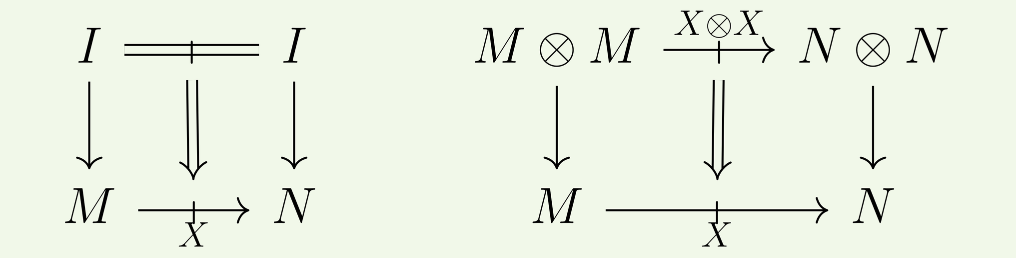
$$\mathsf{strength}_M : T(X,Y) \to T(M \bullet X, M \bullet Y)$$

Being profunctors equipped with a strength in the action of a monoidal category, it is oft-said that Tambara modules are the corresponding notion of profunctor required when generalising from categories to actegories. Indeed in the seminal paper on the topic, Tambara refers to these modules as profunctors with a *two-sided action* [2]. However, in the usual definition there is no object which literally acts on the profunctor in any categorical sense, so we are led to ask if there is some way in which this analogy is made formal.

Here we investigate a notion of 2-dimensional bimodule, showing how Tambara modules arise as a special case while noting that the more general notion allows for useful extensions of certain applications.

(Pseudo)monoids in double categories

Many proarrow equipments have the form of a double category whose loose cells are object-like, with squares being morphisms between them. Viewing monoidal categories as a setting for defining monoid structures on the objects, we extend this perspective, viewing monoidal equipments as a setting for defining monoid structures on the proarrows:

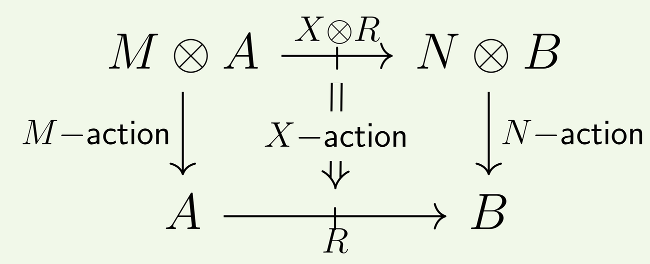


We call X a *monoidal proarrow* when

- M and N are pseudomonoids in the monoidal 2-category $\mathcal{T}(\mathbb{M})$ of tight arrows in $\mathbb{M}$
- X is a monoid object in $\mathbb{M}_1$
- The multiplication and unit cells for X satisfy compatibility equations with the associators and unitors for M and N

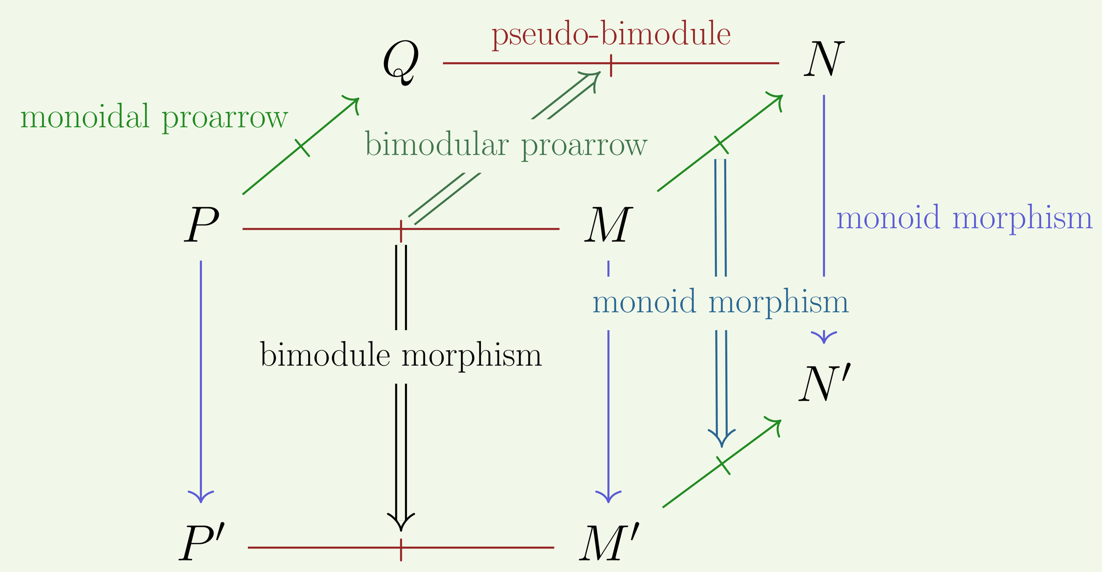
Such structures assemble into a double category $\mathbf{PsMon}(\mathbb{M})$ of pseudomonoids, monoidal proarrows, and monoid morphisms.

Continuing this thread of generalised monoids, we define modules for monoidal proarrows, which are bordered by modules for the corresponding pseudomonoids:



Of course, this notion of module extends to bimodules, assembling into another double category $\mathbf{PsMod}(\mathbb{M})$, of pseudo-bimodules, bimodular proarrows, and bimodule morphisms.

Together, the morphisms of $\mathbf{PsMon}(\mathbb{M})$ and $\mathbf{PsMod}(\mathbb{M})$ assemble into a cubical structure:



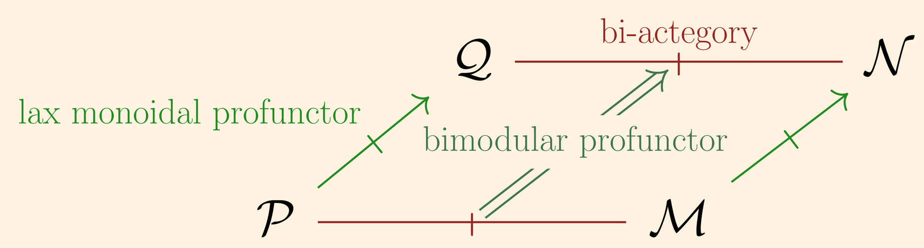
Generalised Tambara Modules[i]

Our motivating example for a triple category of monoidal monads is the case where $\mathbb{M} = \mathbb{P}\mathbf{rof}$. We have that

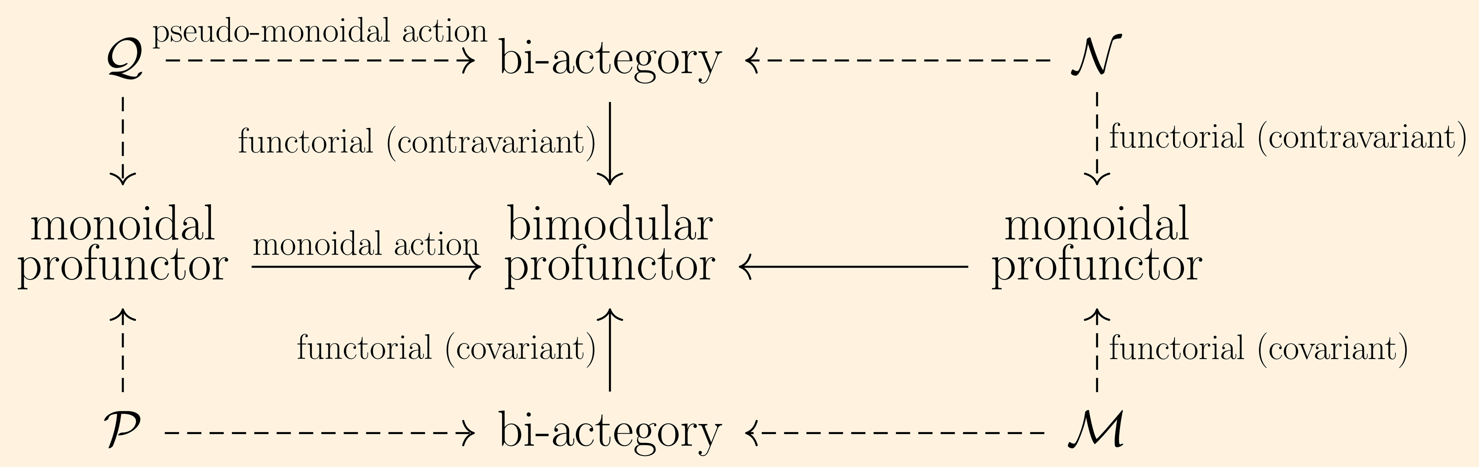
- $\mathbf{PsMon}(\mathbb{P}\mathbf{rof})$ is a double category of monoidal categories, monoidal functors and lax monoidal profunctors.
- The pseudo-bimodules of $\mathbf{PsMod}(\mathbb{P}\mathbf{rof})$ are 2-sided actions of monoidal categories, or *biactegories* [3]:

$$\mathcal{Q} \times \mathcal{C} \to \mathcal{C}$$
$$\mathcal{C} \times \mathcal{N} \to \mathcal{C}$$

In this case, since the proarrows are themselves a kind of bimodule, bimodular proarrows have bimodule structures in two axes:



Here we have two kinds of bimodule: profunctors which are acted on by functoriality, and bi-actegories whose action is from the pseudomonoidal structure. Both kinds of bimodule have further actions on bimodular profunctors which act as two-dimensional bimodules.



The action of a monoidal profunctor F on a bimodular one T amounts to a natural transformation

$$F(Q,P) \times T(X,Y) \xrightarrow{a} T(Q \bullet X, P \bullet Y)$$

but in the case where F is the identity (hom) profunctor this is equivalent to the structure of a strength of T

$$T(X,Y) \xrightarrow{a(1_Q,-)} T(Q \bullet X, Q \bullet Y)$$

Hence we obtain Tambara module as the special case of bimodular profunctors which are globular in one direction.

Application: Generalised Optics[i]

While originally defined for applications in representation theory, Tambara modules have recently found use in the theory of modular data accessors; where they are central in a representation theorem which facilitates their efficient machine encoding in kinded polymorphic languages [4, 5]. The category constructed by this representation theorem, known as a category of *doubles*, or *optics*, can be seen as the result of freely adding counits to a monoidal category[6], modelling bidirectional data flow by a pair of opposed morphisms sharing some state.

The representation theorem holds equally for the generalised form of Tambara modules as actions of a monoidal profunctor, and gives rise to a category of optics whose counits are graded or labelled by elements of the monoidal profunctor in question:

This allows for modelling of optics which coherently carry with them data about the way in which the shared state is exchanged, or which even having differing representations of the data in the forward and backward directions, by situating it in different categories.

Application: Collages of String Diagrams [ii]

Actions of monoidal categories can be seen as giving a way in which monoidal categories interact with eachother. Tambara modules extend this perspective by relating interacting monoidal categories along another axis. We define a notion of a *collage of monoidal actions* as a construction which takes a composable diagram of bimodular profunctors and produces a double category whose string diagrams extend diagrams for monoidal categories by representing several monoidal categories in its multiple regions. This construction gives rise to semantics for functor boxes, diagrams for actegories, and the notion of 'diagrams internal in tubes'. [7]

We conjecture that the collage construction for a diagram of bimodular profunctors $D : \mathbb{I} \to \mathbb{T}\mathbf{amb}$ can be formally defined as a lax colimit of triple functor

$$\mathbf{Col}(D) = \mathrm{colim}_{\mathsf{lax}}(\mathbb{I} \to \mathbb{T}\mathbf{amb} \hookrightarrow \mathbb{D}\mathbf{bl})$$

where $\mathbb{D}\mathbf{bl}$ is a triple category of double categories and double profunctors, studied in upcoming work by Williams [8]

References

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(i): Based on joint work with Matteo Capucci
(ii): Based on joint work with Mario Román